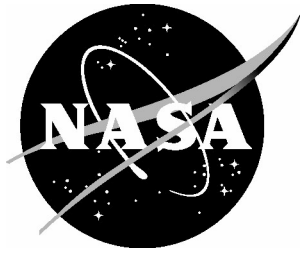


NASA/TM-2007-214872



Quasi-Static Analysis of LaRC THUNDER Actuators

Joel F. Campbell
Langley Research Center, Hampton, Virginia

May 2007

The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the lead center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- **TECHNICAL PUBLICATION.** Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.
- **TECHNICAL MEMORANDUM.** Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.
- **CONTRACTOR REPORT.** Scientific and technical findings by NASA-sponsored contractors and grantees.

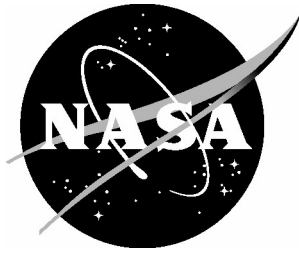
- **CONFERENCE PUBLICATION.** Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.
- **SPECIAL PUBLICATION.** Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.
- **TECHNICAL TRANSLATION.** English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized databases, organizing and publishing research results ... even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at [*http://www.sti.nasa.gov*](http://www.sti.nasa.gov)
- E-mail your question via the Internet to [*help@sti.nasa.gov*](mailto:help@sti.nasa.gov)
- Fax your question to the NASA STI Help Desk at (301) 621-0134
- Phone the NASA STI Help Desk at (301) 621-0390
- Write to:
NASA STI Help Desk
NASA Center for AeroSpace Information
7115 Standard Drive
Hanover, MD 21076-1320

NASA/TM-2007-214872



Quasi-Static Analysis of LaRC THUNDER Actuators

Joel F. Campbell
Langley Research Center, Hampton, Virginia

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

May 2007

Available from:

NASA Center for Aerospace Information (CASI)
7115 Standard Drive
Hanover, MD 21076-1320
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 605-6000

Foreword

This report represents work completed in the 1997-1998 time frame while working as a NASA contractor for SAIC at NASA Langley Research Center. It was originally submitted as a paper to Smart Materials and Structures in Feb., 1998 (Log Number: 00395) and conditionally accepted in July, 1998 pending submission of a minor revision. This was withheld by the author because it was simultaneously submitted as part of a patent disclosure, LAR-15827, in April, 1998.

Much has changed since the time this report was written. This model has since been duplicated and improved upon over the past 9 years since it was written. It is being reproduced here for historical purposes and to make it more referenceable for those who may wish to reference it.

Quasi-Static Analysis of LaRC Thunder Actuators

Joel Campbell
NASA Langley Research Center
Hampton, VA 23681

Abstract

An analytic approach is developed to predict the shape and displacement with voltage in the quasi-static limit of LaRC Thunder Actuators. The problem is treated with classical lamination theory and Von Karman non-linear analysis. In the case of classical lamination theory exact analytic solutions are found. It is shown that classical lamination theory is insufficient to describe the physical situation for large actuators but is sufficient for very small actuators. Numerical results are presented for the non-linear analysis and compared with experimental measurements. Snap-through behavior, bifurcation, and stability are presented and discussed.

Introduction

Thunder actuators are devices constructed from an isotropic laminate of aluminum, LaRC SI adhesive, piezo-electric PZT, and a metal backing of either steel or brass. The LaRC adhesive is a 1 mil thick solid thermoplastic and the aluminum layer is indented to provide electrical conductivity after the actuator is cured at 250 deg C. After the curing process they form a bow shape after reaching room temperature. If simply supported or arranged in a clamshell arrangement they produce a linear motion when an external voltage is applied.

A simple method for controlling the manufacturing process and predicting the actuation amplitude has long been sought for Thunder actuators. A first attempt to do this by an analytic method was completed by BBN [9]. However, this theory treated the problem in 1 dimension with no adhesive layer (a three layer model) and it wasn't clear if this was sufficient to describe the behavior. In particular, there are many variables to be considered. To make assumptions regarding the shape of the device can be misleading. Quite different shapes are possible depending on the geometry and the material properties. The other behavior missing from the BBN model is the snap-through behavior. These devices have two natural bending modes - one in the x and one in the y. In reality, this behavior is a strong function of size. Below some critical size the shape is a perfect dome with equal curvatures but above this critical size the curvatures become unequal.

One approach to this problem is to use classical lamination theory. This is a very old subject based on the theory of composite laminates. With laminates, temperature effects are very important. As a result a very rich theory has been developed to minimize these temperature effects. Although the theory is sufficient to find an optimal lay-up to minimize temperature effects, it is quite deficient in predicting the actual shape in all cases. The reason for this is because it is a linear approximation to a problem that is essentially non-linear. As a result, a non-linear correction to the theory is needed to describe large actuators.

The type of correction used here is one that is valid for large displacement and small strain. This correction is usually attributed to Von Karman [2] in plate theory. Hyer [3] has applied this to his treatment of orthotropic laminates with good success. He uses this in combination with a simple Raleigh-Ritz energy method to obtain solutions exhibiting the snap-through behavior. To apply his theory here, however, it must be reformulated for isotropic laminates, which is a completely separate (and unrelated) case. In particular, the qualitative behavior of orthotropic laminates and isotropic laminates are quite different. Orthotropic laminates produce saddle shapes that approach cylinders when the laminate is large. Isotropic laminates produce domes that approach cylinders when the laminate is large. They are both subsets of the anisotropic case but not subsets of each other. This behavior is well demonstrated with this theory.

Solution by Classical Lamination Theory

Derivation of equations

Classical lamination theory [1] assumes strains are small, and amplitudes are shallow. In addition, it assumes both shearing and extensional strains perpendicular to the surface are zero. These assumptions follow the Kirchhoff-Love hypothesis for shells.

Under deformation the thin plate is described by three variables - u , v and w , where u is the displacement in x direction, v in the y and w is the height of the plate in the z direction. The variables u , v and w are themselves functions of x , y and z . It is assumed u and v follow the relation

$$\begin{aligned} u &= u_0 - z \frac{\partial w_0}{\partial x}, \\ v &= v_0 - z \frac{\partial w_0}{\partial y}, \end{aligned} \tag{1}$$

where the 0 subscript denotes the quantity is measured about the middle surface. The strains are given by [2]

$$\begin{aligned}
\varepsilon_x &= \sqrt{1 + 2 \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \approx \frac{\partial u}{\partial x}, \\
\varepsilon_y &= \sqrt{1 + 2 \frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} - 1 \approx \frac{\partial v}{\partial y}, \\
\gamma_{xy} &= \sin^{-1} \left(\frac{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}}{(1 + \varepsilon_x)(1 + \varepsilon_y)} \right) \approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{aligned} \tag{2}$$

The results of Equation 1 and Equation 2 imply

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2}, \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2}, \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w_0}{\partial x \partial y}.
\end{aligned} \tag{3}$$

The above implies the relationship

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + z \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}, \tag{4}$$

where κ_x , κ_y , and κ_{xy} are the x, y and twist curvatures. The middle surface strains and curvatures are given by,

$$\begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{pmatrix}, \quad (5)$$

$$\begin{pmatrix} \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{pmatrix} = - \begin{pmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial^2 w_0}{\partial x \partial y} \end{pmatrix}.$$

The relationship between stress and strain is given by

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \bar{\mathbf{Q}} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}, \quad (6)$$

where \mathbf{Q} is the reduced stiffness and $\bar{\mathbf{Q}}$ is the transformed reduced stiffness. For isotropic media,

$$\bar{\mathbf{Q}} = \mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{pmatrix} = \begin{pmatrix} \frac{Y}{1-\nu^2} & \frac{\nu Y}{1-\nu^2} & 0 \\ \frac{\nu Y}{1-\nu^2} & \frac{Y}{1-\nu^2} & 0 \\ 0 & 0 & \frac{Y}{2(1+\nu)} \end{pmatrix}, \quad (7)$$

where Y is Young's modulus and ν is Poisson's ratio. To find the force and moments we first find the stresses for the k 'th layer by,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}_k = (\bar{\mathbf{Q}})_k \begin{bmatrix} \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + z \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} \end{bmatrix}. \quad (8)$$

Now we use the relationship

$$\begin{pmatrix} N_x^0 \\ N_y^0 \\ N_{xy}^0 \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} dz, \quad (9)$$

$$\begin{pmatrix} M_x^0 \\ M_y^0 \\ M_{xy}^0 \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} z dz,$$

where h is the thickness. By Equation 9 and 8 we find,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + \mathbf{B} \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}, \quad (10)$$

$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \mathbf{B} \begin{pmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} + \mathbf{D} \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix},$$

where \mathbf{A} , \mathbf{B} , and \mathbf{D} are the extensional, coupling and bending stiffness matrices and are given by,

$$A_{ij} = \sum_{k=1}^N (Q_{ij})_k (z_k - z_{k-1}),$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N (Q_{ij})_k (z_k^2 - z_{k-1}^2), \quad (11)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N (Q_{ij})_k (z_k^3 - z_{k-1}^3),$$

where the z 's are measured with respect to $z=0$ (at a point half way through the thickness). Since each layer is isotropic the symmetry of the stiffness matrices will be the same as the reduced stiffness matrix. Moreover, since in this case the layers are not symmetric about $z=0$ the coupling stiffness will be non-zero so that,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & A_1 & 0 \\ 0 & 0 & A_3 \end{pmatrix}, \\
\mathbf{B} &= \begin{pmatrix} B_1 & B_2 & 0 \\ B_2 & B_1 & 0 \\ 0 & 0 & B_3 \end{pmatrix}, \\
\mathbf{D} &= \begin{pmatrix} D_1 & D_2 & 0 \\ D_2 & D_1 & 0 \\ 0 & 0 & D_3 \end{pmatrix}
\end{aligned} \tag{12}$$

We now have everything we need to find the thermal and actuation loading so we may solve the problem.

Thermal Loading

The thermal loading takes two forms. The first is extensional and the second is bending. The problem is greatly simplified since we have only isotropic layers with equal thermal strains in each direction. The thermal loading may be calculated by,

$$\begin{aligned}
\begin{pmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{pmatrix} &= \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \Delta T \, dz \equiv \begin{pmatrix} N_x^T \\ N_y^T \\ 0 \end{pmatrix}, \\
\begin{pmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{pmatrix} &= \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \Delta T \, z \, dz \equiv \begin{pmatrix} M_x^T \\ M_y^T \\ 0 \end{pmatrix},
\end{aligned} \tag{13}$$

where,

$$\begin{aligned}
N^T &= \Delta T \sum_{k=1}^N \alpha_k (\mathcal{Q}_1 + \mathcal{Q}_2)_k (z_k - z_{k-1}), \\
M^T &= \frac{\Delta T}{2} \sum_{k=1}^N \alpha_k (\mathcal{Q}_1 + \mathcal{Q}_2)_k (z_k^2 - z_{k-1}^2).
\end{aligned} \tag{14}$$

Actuation Loading

By analogy with thermal loading,

$$\begin{pmatrix} N_x^A \\ N_y^A \\ N_{xy}^A \end{pmatrix} = \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} d_{31} \\ d_{31} \\ 0 \end{pmatrix} \frac{V}{t} dz \equiv \begin{pmatrix} N^A \\ N^A \\ 0 \end{pmatrix}. \quad (15)$$

$$\begin{pmatrix} M_x^A \\ M_y^A \\ M_{xy}^A \end{pmatrix} = \int_{-h/2}^{h/2} \mathbf{Q} \begin{pmatrix} d_{31} \\ d_{31} \\ 0 \end{pmatrix} \frac{V}{t} z dz \equiv \begin{pmatrix} M^A \\ M^A \\ 0 \end{pmatrix},$$

where t is the thickness of the PZT, V is the applied voltage, d_{31} is the actuation constant in a plane perpendicular to the poling axis and,

$$N^A = d_{31} (Q_1 + Q_2)_3 V,$$

$$M^A = d_{31} (Q_1 + Q_2)_3 (z_3 + z_2) V. \quad (16)$$

Here it is assumed we only have actuation on the third layer.

Thermal and Actuation Displacement

To find the resulting strains and curvatures in terms of the actuation loading we first add both the thermal and actuation sources as

$$\begin{pmatrix} N'_x \\ N'_y \\ N'_{xy} \end{pmatrix} \equiv \begin{pmatrix} N^A + N^T \\ N^A + N^T \\ 0 \end{pmatrix} = \begin{pmatrix} N' \\ N' \\ 0 \end{pmatrix}, \quad (17)$$

$$\begin{pmatrix} M'_x \\ M'_y \\ M'_{xy} \end{pmatrix} \equiv \begin{pmatrix} M^A + M^T \\ M^A + M^T \\ 0 \end{pmatrix} = \begin{pmatrix} M' \\ M' \\ 0 \end{pmatrix}.$$

Now we use the results of Equation 10 so that

$$\begin{pmatrix} N \\ N \\ 0 \\ M \\ M \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}. \quad (18)$$

Inversion of the above gives

$$\begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{pmatrix}^{-1} \begin{pmatrix} N \\ N \\ 0 \\ M \\ M \\ 0 \end{pmatrix}. \quad (19)$$

By using the results of 12 and inverting the 6x6 matrix we find,

$$\begin{aligned} \varepsilon_x^0 \equiv \varepsilon_y^0 &= \frac{(D_1 + D_2) N - (B_1 + B_2) M}{(A_1 + A_2)(D_1 + D_2) - (B_1 + B_2)^2}, \\ \kappa_x \equiv \kappa_y &= \frac{(A_1 + A_2) M - (B_1 + B_2) N}{(A_1 + A_2)(D_1 + D_2) - (B_1 + B_2)^2}, \\ \gamma_{xy} &= \kappa_{xy} = 0. \end{aligned} \quad (20)$$

So we have thus found the curvature in terms of known quantities. The shape is that of a dome. The simplicity of the relationship comes about because the layers are isotropic. The solution is valid for small actuators such as the round ones used for pumps and speakers.

Solution by the Von Karman Non-Linear Approximation

The Von Karman approximation assumes large displacement and small strain. This suggests including second order terms in w in addition to the linear terms for u , v and w . Expanding Equation 2 to include these terms give,

$$\begin{aligned}
\varepsilon_x &\approx \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \\
\varepsilon_y &\approx \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \\
\gamma_{xy} &\approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}.
\end{aligned} \tag{21}$$

This combined with the results of Equation 1 give,

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial^2 w_0}{\partial x^2}, \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 - z \frac{\partial^2 w_0}{\partial y^2}, \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} - 2z \frac{\partial^2 w_0}{\partial x \partial y}.
\end{aligned} \tag{22}$$

The total energy takes the form,

$$U = \iiint (dU_0 - dW_T - dW_A), \tag{23}$$

where dU_1 is the stored elastic energy volume density and dW_T and dW_A are the temperature and actuation contributions. The elastic energy portion of it for the k 'th layer is,

$$\begin{aligned}
(dU_0)_k &= \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}), \\
&= \left(\frac{1}{2} Q_1 \varepsilon_x^2 + Q_2 \varepsilon_x \varepsilon_y + \frac{1}{2} Q_1 \varepsilon_y^2 + \frac{1}{2} Q_3 \gamma_{xy}^2 \right)_k.
\end{aligned} \tag{24}$$

For the thermal portion we have,

$$\begin{aligned}
(dW_T)_k &= (\sigma_x^T \varepsilon_x + \sigma_y^T \varepsilon_y)_k, \\
&= \left((Q_1 + Q_2) \alpha \Delta T \varepsilon_x + (Q_1 + Q_2) \alpha \Delta T \varepsilon_y \right)_k.
\end{aligned} \tag{25}$$

For the actuation portion we have

$$\begin{aligned} (dW_A)_3 &= (\sigma_x^A \epsilon_x + \sigma_y^A \epsilon_y)_3, \\ &= (Q_1 + Q_2)_3 d_{31} \frac{V}{t} \epsilon_x + (Q_1 + Q_2)_3 d_{31} \frac{V}{t} \epsilon_y \end{aligned} \quad (26)$$

Now that we have the general form for the energy integral we may try a Raleigh-Ritz type solution. The method involves making a guess for the solution. This guess should look qualitatively like the expected solution yet be flexible enough to allow for adjustment. This adjustment comes in the form of changing the guess based on a finite number of parameters that the guess contains. The parameters are adjusted in such a way as to minimize the total energy of the system. In this situation it would be advantageous to choose a guess that is close enough to the classical lamination result that in the linear limit (small scaling) the solution approaches the classical lamination result. Such a guess takes the form

$$w^0 = \frac{1}{2} (a x^2 + b y^2). \quad (27)$$

Classical lamination theory assumes no shearing strain between layers for thermal expansion. If we make the same assumptions here we find u^0 and v^0 must take the form,

$$\begin{aligned} u^0 &= c x - \frac{1}{6} a^2 x^3 - \frac{1}{4} a b x y^2, \\ v^0 &= d y - \frac{1}{6} a^2 y^3 - \frac{1}{4} a b x^2 y. \end{aligned} \quad (28)$$

The ansatz represented by Equation 27 and 28 has been used successfully by Hyer [3] in his treatment of orthotropic laminates. It is sufficiently flexible enough to be used with isotropic laminates as well as we will soon see. Equations 22, 27 and 28 give,

$$\begin{aligned} \epsilon_x &= c - \frac{1}{4} a b y^2 - a z, \\ \epsilon_y &= d - \frac{1}{4} a b x^2 - b z, \\ \gamma_{xy} &= 0. \end{aligned} \quad (29)$$

All that is needed now is to use Equation 23 with Equations 24, 25 and 26 and 29 to find the total energy. The solution is obtained by minimizing the total energy with respect to a, b, c and d. The result of the minimization is

$$\begin{aligned}
& D_1 a + K_8 b - B_1 c - B_2 d + K_1 b^2 + K_6 a b - K_4 b c - K_3 b d + \\
& K_7 a b^2 + M' = 0, \\
& K_8 a + D_1 b - B_2 c - B_1 d + K_2 a^2 + K_5 a b - K_4 a c - K_3 a d + \\
& K_7 a^2 b + M' = 0, \\
& -B_1 a - B_2 b + A_1 c + A_2 d - K_4 a b - N' = 0, \\
& -B_2 a - B_1 b + A_2 c + A_1 d - K_3 a b - N' = 0,
\end{aligned} \tag{30}$$

where,

$$\begin{aligned}
K_1 &= \frac{1}{48} (B_1 L_x^2 + B_2 L_y^2) \\
K_2 &= \frac{1}{48} (B_2 L_x^2 + B_1 L_y^2) \\
K_3 &= \frac{1}{48} (A_1 L_x^2 + A_2 L_y^2), \\
K_4 &= \frac{1}{48} (A_2 L_x^2 + A_1 L_y^2), \\
K_5 &= \frac{1}{24} (B_1 L_x^2 + B_2 L_y^2), \\
K_6 &= \frac{1}{24} (B_2 L_x^2 + B_1 L_y^2), \\
K_7 &= \frac{A_2 L_x^2 L_y^2}{1152} + \frac{A_1 (L_x^4 + L_y^4)}{1280}, \\
K_8 &= D_2 + \frac{N'}{48} (L_x^2 + L_y^2).
\end{aligned} \tag{31}$$

The A's, B's and D's have the same meaning as before and N' is the same as well. The lengths L_x and L_y are the length and width of the actuator. It is a simple matter to show that the solution in the linear limit (small scaling - $L_x=L_y=0$) that the solution to these equations matches the classical lamination result if we take $a=-\kappa_x$, $b=-\kappa_y$, $c=\epsilon_x$ and $d=\epsilon_y$. The non-linear equations are coupled and third order so it is unlikely that an exact analytic solution exists. It may be possible to find close approximate solutions to them, however. In any event they may be solved numerically with no difficulty.

Unlike the classical lamination result the non-linear analysis predicts unequal curvatures that depend on the magnitude of the scaling and the aspect ratio. It also predicts multiple possible solutions. The nature of these solutions are dome-like solutions which approach cylinders in the limit of large scaling.

As a comparison for a particular case we compute a simple example. For this case 1 mil aluminum, 1 mil LaRC Si, 5 mil PZT and 2 mil stainless was chosen.

The non-linear analysis predicts a total of 5 possible solutions. Out of the five, 3 are non-physical so we are left with two real solutions. The snap-through behavior is clearly exhibited in this case. As a comparison this is compared with classical lamination theory. The behavior is quite different in this case. It should be noted that the non-linear theory may admit very different solutions depending on the geometry, aspect ratio, etc.

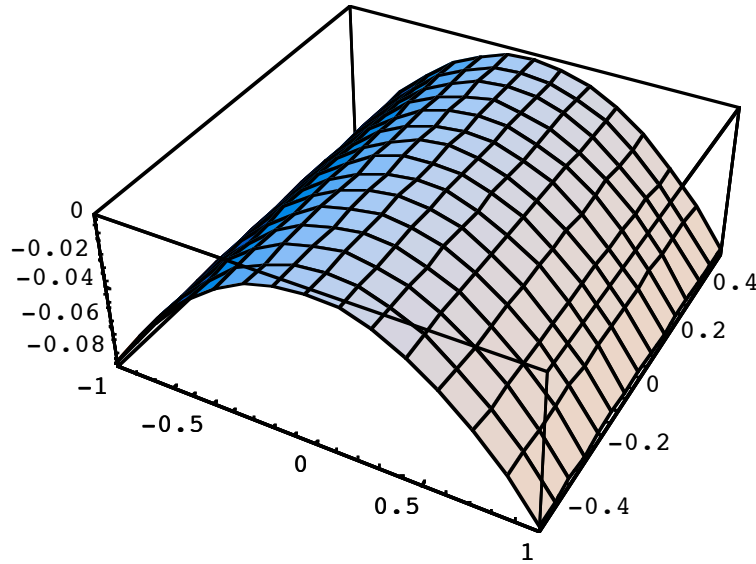


Figure 1. Non-linear Solution #1

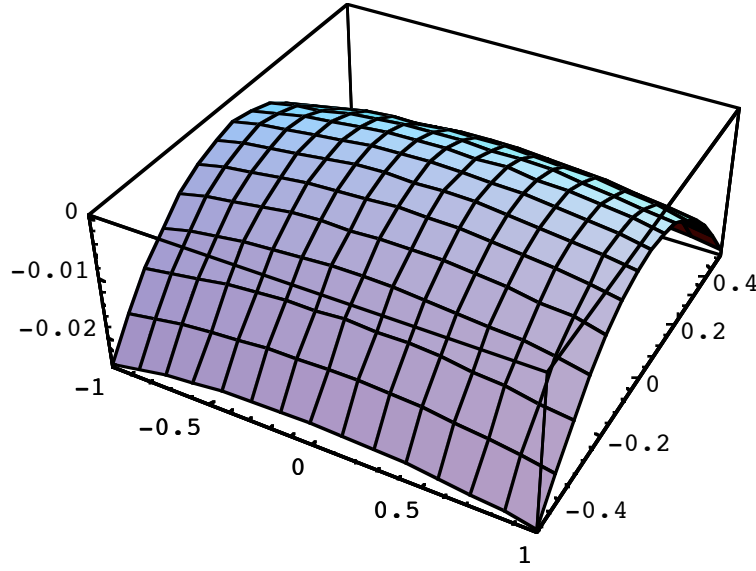


Figure 2. Non-linear Solution #2

Stability

The stability of each solution is determined by whether or not the solutions are maxima or minima. If a particular solution represents a minimum then it is stable. If it is a maximum, it is unstable. The total energy of the system may be viewed as a function of several variables. As such, once an extremum is found, the test whether it represents minima is determined by whether the determinant of F is positive definite where

$$\begin{aligned} \text{Det}[F] &> 0, \\ F_{ij} &= \frac{\partial^2 U}{\partial x_i \partial x_j}. \end{aligned} \tag{32}$$

Fortunately $\partial U / \partial x_i$ is represented by the left hand side of Equation 30 (to within a multiplicative factor) so all we have to do are the x_j differentiations to find the matrix elements.

Bifurcation

As stated earlier, The solution may take on different shapes depending on the scaling. To demonstrate this, numerical results were completed for a square piece having the same material properties as before except we vary the size from $x=0$ to $x=2$ in. For a small piece the shape is a dome as predicted by classical lamination theory. At some critical size the shape turns suddenly into a dome with unequal curvatures. In the limit of large scaling it becomes a cylinder. Figure 3 is a plot of both the x and y curvatures. In this case the critical size is at approximately 1 inch square. This is quite different from the results of Hyer[3], who found saddle solutions. However, this is easily explained by the

fact that his work was with orthotropic laminates as opposed to isotropic laminates. In the case of orthotropic laminates even classical lamination theory predicts a saddle.

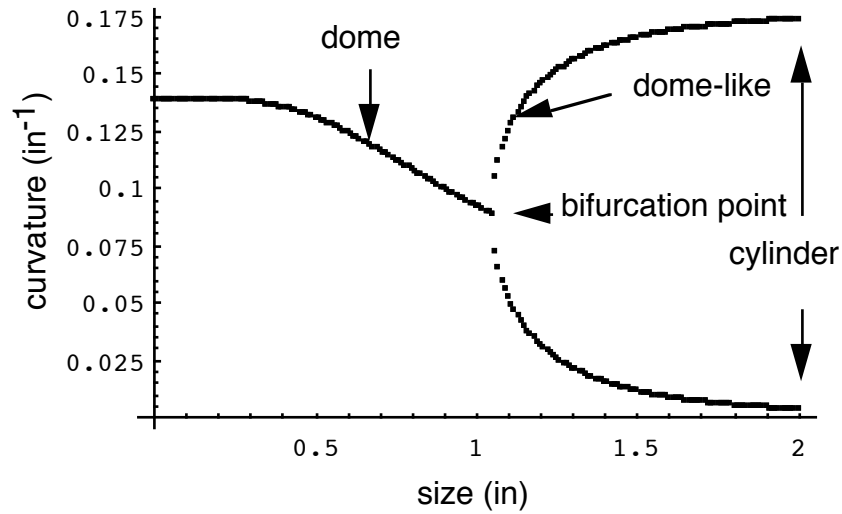


Figure 3. Bifurcation plot of x and y curvature as a function of size for a square actuator. Stable solutions only.

Preliminary calculations seem to indicate the currently manufactured devices are close enough to the bifurcation point for these effects to be important.

Experimental Results

As an example we compare the model with known measured data, (assumed) material properties, and thicknesses. A table of these samples is shown below.

Sample 1 (2"x1").

	Material	Thickness	Y (GPa)	CTE($10^{-6}/\text{deg C}$)
Layer 1	Aluminum	.001"	73	22
Layer 2	LaRC Si	.001"(irregular)	4	46
Layer 3	PZT 5a	.01"	63	1
Layer 4	LaRC Si	.001"(irregular)	4	46
Layer 5	stainless steel	.005"	200	16

Sample 2 (2"x1").

	Material	Thickness	Y (GPa)	CTE($10^{-6}/\text{deg C}$)
Layer 1	Aluminum	.001"	73	22
Layer 2	LaRC Si	.001"(irregular)	4	46

Layer 3	PZT 5a	.01"	63	1
Layer 4	LaRC Si	.001"(irregular)	4	46
Layer 5	stainless steel	.003"	200	16

Sample 3 (2"x1").

	Material	Thickness	Y (GPa)	CTE(10^{-6} /deg C)
Layer 1	Aluminum	.001"	73	22
Layer 2	LaRC Si	.001"(irregular)	4	46
Layer 3	PZT 5a	.01"	63	1
Layer 4	LaRC Si	.001"(irregular)	4	46
Layer 5	stainless steel	.001"	200	16

Sample 4 (3"x1.75").

	Material	Thickness	Y (GPa)	CTE(10^{-6} /deg C)
Layer 1	Aluminum	.001"	73	22
Layer 2	LaRC Si	.001"(irregular)	4	46
Layer 3	PZT 5a	.0078"	63	1
Layer 4	LaRC Si	.001"(irregular)	4	46
Layer 5	1020 steel	.005"	200	16

Sample 5 (3"x1.75").

	Material	Thickness	Y (GPa)	CTE(10^{-6} /deg C)
Layer 1	Aluminum	.001"	73	22
Layer 2	LaRC Si	.001"(irregular)	4	46
Layer 3	PZT 5a	.0078"	63	1
Layer 4	LaRC Si	.001"(irregular)	4	46
Layer 5	1020 steel	.003"	200	16

Sample 6 (3"x1.75").

	Material	Thickness	Y (GPa)	CTE(10^{-6} /deg C)
Layer 1	Aluminum	.001"	73	22
Layer 2	LaRC Si	.001"(irregular)	4	46
Layer 3	PZT 5a	.0078"	63	1
Layer 4	LaRC Si	.001"(irregular)	4	46
Layer 5	1020 steel	.001"	200	16

We compare the measured dome height with the predicted values. The first 3 samples were taken from the BBN report and the last 3 were measured at LaRC. The setting temperature of LaRC SI was assumed to be 250 deg C for the calculation.

Table 1. Measured vs. Predicted dome heights

Sample	1	2	3	4	5	6
Measured	.114"	.126"	.193"	.198"	.281"	.466"
Predicted	.116"	.103"	.029"	.29"	.282"	.113
Sol. Type	cylinder	cylinder	dome	cylinder	cylinder	cylinder
BBN pred.	.098"	.095"	.047"	*	*	*

The results of Table [1] show that the model is fairly accurate except when the base layer has a small thickness. This would suggest the adhesive layer irregularities have an effect on the solution. In the case of the smaller BBN 1 mil sample a transition from cylinder to dome was predicted. This would suggest the solution is close to the bifurcation point. Since any small error near the bifurcation point produces huge differences in the curvatures, this could be another explanation for the discrepancy for a small base layer. A third explanation could be the second order non-linear correction may not be enough to accurately describe the real physical shape in all cases. It is interesting the BBN model predicts the same trend. The reasons for this are not clear since they do not consider the adhesive layer at all.

Conclusions

More work needs to be done to accurately describe the problem to take into account irregularities in the adhesive layers. These irregularities are currently part of the manufacturing process and are currently needed to provide electrical conductivity for the device. At least in the case of the top layer it may be possible to treat the aluminum/adhesive combination as a single layer with special material properties of it's own . In the case of the steel/adhesive boundary, detailed measurements need to be taken to see if there is some trend in the irregularities. If there is it may be possible to model it. In any event, the model appears to be very accurate as it is for a base layer of 3 mils and above.

This model shows some very interesting behavior and is a giant leap forward in demonstrating the snap-through behavior. It also gives insight into how the devices should be manufactured. It shows that in some cases perfect domes are possible and that one should construct the devices as far from the bifurcation point as possible so that the device is in the cylinder range.

Numerical work needs to be done to show the voltage effects in Equation 16. In addition, external loading effects need to be taken into account. This is a very straightforward problem using the energy approach and would result in a slight variation of the Equation 30. Several variations of this are possible. The first would be to find the change in amplitude with voltage as a function of load. Another more interesting case would be to constrain the amplitude and allow the load to vary. This would result in an equation for blocked force as a function of constrained distance.

Acknowledgements

I would like to thank Professor D. S. Cairns of Montana State University for his many very helpful comments and suggestions. In addition, I would like to thank Bill Hunter, Stephanie Wise, Tony Jalink, Bob Copeland, Robert Fox and Richard Hellbaum (all of LaRC) for helping provide measurements and other information.

References

1. Mechanics of Composite Materials
R. M. Jones
McGraw-Hill 1975
2. Anisotropic Plates
S. G. Lekhnitskii
Gordon and Breach Publishers, 1968
3. M. W. Hyer
Calculations of the Room-Temperature Shapes of Unsymmetric Laminates
J. Composite Materials
Vol. 15, July 1981
4. A. Hamamoto and M. W. Hyer
Non-Linear Temperature-Curvature Relationships for
Unsymmetric Graphite-Epoxy Laminates
Int. J. Solid Structures
Vol 23, No.7, pp 919-935, 1987
5. M. W. Hyer and P. C. Bhavani
Suppression of Anticlastic Curvature in Isotropic and Composite Plates
Int. J. Solid Structures
Vol. 20, No. 6, pp 553-570, 1987
6. R. Yang and M. A. Bhatti
Non-linear Static and Dynamic Analysis of Plates
Journal of Engineering Mechanics
Vol. 111, No. 2, Feb., 1985
7. D. A. Pecknold and J. Ghaboussi
Snap-Through and Bifurcation in a Simple Structure
Journal of Engineering Mechanics
Vol. 111, No. 2, Feb., 1985

8. J. F. Campbell and D. S. Cairns
A Multi-Purpose Sensor for Composite
Laminates Based on a Piezo-Electric Film
J. Composite Materials
Vol. 26, No.3 /1992
9. Alan R.D. Curtis
A Model of the THUNDER Actuator
BBN Tech Memo #1190
NASA Contract NAS 1 20101, Dec. 31, 1997
10. Morgan Matroc Limited
Piezoelectric Ceramic Products (product manual)

REPORT DOCUMENTATION PAGE					Form Approved OMB No. 0704-0188	
<p>The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.</p> <p>PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.</p>						
1. REPORT DATE (DD-MM-YYYY)		2. REPORT TYPE			3. DATES COVERED (From - To)	
01-05 - 2007		Technical Memorandum				
4. TITLE AND SUBTITLE Quasi-Static Analysis of LaRC THUNDER Actuators				5a. CONTRACT NUMBER		
				5b. GRANT NUMBER		
				5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Campbell, Joel F.				5d. PROJECT NUMBER		
				5e. TASK NUMBER		
				5f. WORK UNIT NUMBER 810031.07.03		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center Hampton, VA 23681-2199				8. PERFORMING ORGANIZATION REPORT NUMBER L-19355		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001				10. SPONSOR/MONITOR'S ACRONYM(S) NASA		
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) NASA/TM-2007-214872		
12. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited Subject Category 37 Availability: NASA CASI (301) 621-0390						
13. SUPPLEMENTARY NOTES An electronic version can be found at http://ntrs.nasa.gov						
14. ABSTRACT An analytic approach is developed to predict the shape and displacement with voltage in the quasi-static limit of LaRC Thunder Actuators. The problem is treated with classical lamination theory and Von Karman non-linear analysis. In the case of classical lamination theory exact analytic solutions are found. It is shown that classical lamination theory is insufficient to describe the physical situation for large actuators but is sufficient for very small actuators. Numerical results are presented for the non-linear analysis and compared with experimental measurements. Snap-through behavior, bifurcation, and stability are presented and discussed.						
15. SUBJECT TERMS PZT; THUNDER; Von Karman; Actuator; Bifurcation; LaRC-Si; Nonlinear; Piezo-electric; Rayleigh-Ritz; Snap-through						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON	
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Help Desk (email: help@sti.nasa.gov)	
U	U	U	UU	24	19b. TELEPHONE NUMBER (Include area code) (301) 621-0390	